ON EQUATION FOR INITIAL VALUES IN THEORY OF THE SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

We consider the properties of the second order nonlinear differential equations $b'' = g(a, b, b')$ with the function $g(a, b, b' = c)$ satisfying the following nonlinear partial differential equation

$$g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bbcc} + 2cgg_{bccc}$$

$$+g^2 g_{cccc} + (g_a + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bbc} - cg_c g_{bcc}$$

$$-3gg_{bcc} - g_c g_{acc} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0\,.$$

Any equation $b'' = g(a, b, b')$ with this condition on function $g(a, b, b')$ has the General Integral $F(a, b, x, y) = 0$ shared with General Integral of the second order ODE's $y'' = f(x, y, y')$ with condition $\frac{\partial^4 f}{\partial y'^4} = 0$ on function $f(x, y, y')$ or

$$y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0$$

with some coefficients $a_i(x, y)$.

1 Introduction

The relation between the equations in form

$$y'' + a_1(x, y)y'^3 + 3a_2(x, y)y'^2 + 3a_3(x, y)y' + a_4(x, y) = 0 \tag{1}$$

and

$$b'' = g(a, b, b') \tag{2}$$

with function $g(a, b, b')$ satisfying the p.d.e

$$g_{aacc} + 2cg_{abcc} + 2gg_{accc} + c^2 g_{bbcc} + 2cgg_{bccc}+$$

$$g^2 g_{cccc} + (g_a + cg_b)g_{ccc} - 4g_{abc} - 4cg_{bbc} - cg_c g_{bcc}- \tag{3}$$

$$3gg_{bcc} - g_c g_{acc} + 4g_c g_{bc} - 3g_b g_{cc} + 6g_{bb} = 0.$$

from geometrical point of view was studied by E.Cartan [1].

In fact, according to the expressions on curvature of the space of linear elements (x,y,y') connected with equation (1)

$$\Omega_2^1 = a[\omega^2 \wedge \omega_1^2], \quad \Omega_1^0 = b[\omega^1 \wedge \omega^2], \quad \Omega_2^0 = h[\omega^1 \wedge \omega^2] + k[\omega^2 \wedge \omega_1^2].$$

where:

$$a = -\frac{1}{6}\frac{\partial^4 f}{\partial y'^4}, \quad h = \frac{\partial b}{\partial y'}, \quad k = -\frac{\partial \mu}{\partial y'} - \frac{1}{6}\frac{\partial^2 f}{\partial^2 y'}\frac{\partial^3 f}{\partial^3 y'},$$

and

$$\begin{aligned} 6b &= f_{xxy'y'} + 2y' f_{xyy'y'} + 2f f_{xy'y'y'} + y'^2 f_{yyy'y'} + 2y' f f_{yy'y'y'} \\ &+ f^2 f_{y'y'y'y'} + (f_x + y' f_y) f_{y'y'y'} - 4f_{xyy'} - 4y' f_{yyy'} - y' f_{y'} f_{yy'y'} \\ &- 3f f_{yy'y'} - f_{y'} f_{xy'y'} + 4f_{y'} f_{yy'} - 3f_y f_{y'y'} + 6f_{yy}. \end{aligned}$$

two types of equations by a natural way are evolved: the first type from the condition $a = 0$ and second type from the condition $b = 0$.

The first condition $a = 0$ the equation in form (1) is detemined and the second condition lead to the equations (2) where the function $g(a, b, b')$ satisfies the above p.d.e. (3).

From the elementary point of view the relation between both equations (1) and (2) is a result of the special properties of their General Integral

$$F(x, y, a, b) = 0.$$

So we have the following fundamental diagramm:

$$\begin{array}{ccccc} & & F(x,y,a,b) = 0 & & \\ & \swarrow\nearrow & & \searrow\nwarrow & \\ y'' = f(x,y,y') & & & & b'' = g(a,b,b') \\ \Updownarrow & & & & \Updownarrow \\ M^3(x,y,y') & & \Longleftrightarrow & & N^3(a,b,b') \end{array}$$

which is presented the General Integral $F(x, y, a, b) = 0$ (as some 3-dim orbifold) in form of the twice nontrivial fibre bundles on circles over corresponding surfaces:

$$M^3(x, y, y') = U^2(x, y) \times S^1 \quad \text{and} \quad N^3(a, b, b') = V^2(a, b) \times S^1.$$

2 An examples of solutions of dual equation

Let us consider the solutions of equation (3).

It has many types of reductions and the symplest of them are

$$g = c^\alpha \omega[ac^{\alpha-1}], \quad g = c^\alpha \omega[bc^{\alpha-2}], \quad g = c^\alpha \omega[ac^{\alpha-1}, bc^{\alpha-2}], \quad g = a^{-\alpha}\omega[ca^{\alpha-1}],$$

$$g = b^{1-2\alpha}\omega[cb^{\alpha-1}], \quad g = a^{-1}\omega(c - b/a), \quad g = a^{-3}\omega[b/a, b - ac], \quad g = a^{\beta/\alpha-2}\omega[b^\alpha/a^\beta, c^\alpha/a^{\beta-\alpha}].$$

For any type of reduction we can write corresponding equation (2) and then integrate it.

As example for the function

$$g = a^{-\gamma}A(ca^{\gamma-1})$$

we get the equation

$$[A + (\gamma-1)\xi]^2 A^{1V} + 3(\gamma-2)[A + (\gamma-1)\xi]A^{111} + (2-\gamma)A^1 A^{11} + (\gamma^2 - 5\gamma + 6)A^{11} = 0.$$

One solution of this equation is

$$A - (2-\gamma)[\xi(1+\xi^2) + (1+\xi^2)^{3/2}] + (1-\gamma)\xi$$

This solution is coresponded to the equation

$$b'' = \frac{1}{a}[b'(1+b'^2) + (1+b'^2)^{3/2}]$$

with Gneral Integral

$$F(x,y,a,b) = (y+b)^2 + a^2 - 2ax = 0$$

The dual equation has the form

$$y'' = -\frac{1}{2x}(y'^3 + y')$$

Remark that the first examples of solutions of equation (3) was obtained in [6-9].

The

Proposition 1 *Equation (3) can be represent in form*

$$\begin{aligned} g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b &= h(a,b,c), \\ h_{ac} + gh_{cc} - g_c h_c + ch_{bc} - 3h_b &= 0. \end{aligned} \tag{4}$$

From this is followed that exists the class of equations (2) with function $g(a,b,c)$ satisfying the condition

$$g_{ac} + gg_{cc} - g_c^2/2 + cg_{bc} - 2g_b = 0. \tag{5}$$

which is more readily solved then equation (3).

Here we present some solutions of the equation (8) as function depending on two variables $g = g(a,c)$

In case when $g = g(a,c)$ and $h = 0$ we have the equation

$$g_{ac} + gg_{cc} - \frac{1}{2}{g_c}^2 = 0\,.$$

To integrate this equation we can transform its in more convenient form using variable $g_c = f(a,c)$. Then one obtains:

$$2f_c f_{ac} + (f^2 - 2f_a)f_{cc} = 0\,.$$

After the Legendre-transformation we obtain the equation:

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0\,.$$

Using the new variable $\xi\omega_\xi + \eta\omega_\eta - \omega = R$ we have the new equation for R:

$$R_\xi - \frac{1}{2}R^2\omega_{\xi\xi} = 0$$

and the following relations:

$$\omega_\eta = \frac{\omega}{\eta} + \frac{R}{\eta} + \frac{2\xi}{\eta R} - \frac{\xi A(\eta)}{\eta}\,,$$

$$\omega_\xi = -\frac{2}{R} + A(\eta)$$

with arbitrary function $A(\eta)$. From the conditions of compatibility is followed:

$$2\eta R_\eta + R_\xi(2\xi - R^2) + \eta A_\eta R^2 = 0\,.$$

Integrating this equation we can obtain general integral.

In the particular case: $A = \frac{1}{\eta}$ we have:

$$\frac{R^2}{R - 2\eta} = -\frac{\xi}{\eta} + \Phi(\frac{1}{\eta} - \frac{2}{R})\,.$$

At the condition $A = 0$ we obtain the equation:

$$2\eta R_\eta + (2\xi - R^2)R_\xi = 0\,,$$

which has the solution:

$$R^2 = 2\xi + 2\eta\Phi(R)\,,$$

were $\Phi(R)$ is arbitrary function.

After choosing the function $\Phi(R)$ we can find the function ω and then using the inverse Legendre transformation the function g which is determined dual equation $b'' = g(a, c)$.

Remark 1 *The solutions of the equations of type*

$$u_{xy} = uu_{xx} + \epsilon u_x^2$$

was constructed in [19]. In work of [20] was showed that they can be present in form

$$u = B'(y) + \int [A(z) - \epsilon y]^{(1-\epsilon)/\epsilon} dz,$$

$$x = -B(y) + \int [A(z) - \epsilon y]^{1/\epsilon} dz.$$

To integrate above equations we apply the parametric representation

$$g = A(a) + U(a,\tau), \quad c = B(a) + V(a,\tau). \tag{11}$$

Using the formulas

$$g_c = \frac{g_\tau}{c_\tau}, \quad g_a = g_a + g_\tau \tau_a$$

we get after the substitution in (10) the conditions

$$A(a) = \frac{dB}{da}$$

and

$$U_{a\tau} - \left(\frac{V_a U_\tau}{V_\tau}\right)_\tau + U\left(\frac{U_\tau}{V_\tau}\right)_\tau - \frac{1}{2}\frac{U_\tau^2}{V_\tau} = 0.$$

So we get one equation for two functions $U(a,\tau)$ and $V(a,\tau)$. Any solution of this equation are determined the solution of equation (10) in form (11).

Let us consider the examples.

$$A = B = 0, \quad U = 2\tau - \frac{a\tau^2}{2}, \quad V = a\tau - 2\ln(\tau)$$

Using the representation

$$U = \tau\omega_\tau - \omega, \quad V = \omega_\tau$$

it is possible to obtain others solutions of this equation.

Equation

$$g_{ac} = gg_{cc} - \frac{1}{2}{g_c}^2 .$$

can be integrate in explicite form and solutions are

$$g = -H'(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^3} ,$$

$$c = H(a) + \int \frac{dz}{[A(z) + \frac{1}{2}a]^2} ,$$

with arbitrary functions $H(a)$ and $A(z)$.

In fact, for $A(z) = z$ we have

$$g = -H'(a) + \int \frac{dz}{[z) + \frac{1}{2}a]^3} = -H'(a) - \frac{1}{2}\frac{1}{[z) + \frac{1}{2}a]^2} ,$$

and

$$c = H(a) + \int \frac{dz}{[z + \frac{1}{2}a]^2} = H(a) - \frac{1}{[z) + \frac{1}{2}a]^3} ,$$

As result we get the solution

Remark 2 *In general case the equation*

$$g_{acc} + gg_{ccc} = 0,$$

is equivalent the equation

$$g_{ac} + gg_{cc} - \frac{1}{2}{g_c}^2 = B(a) .$$

It can be intgrate with help of Legender- transformation as in previous case.

Realy, we get

$$[(\xi\omega_\xi + \eta\omega_\eta - \omega)^2 - 2\xi + 2B(\omega_\xi)]\omega_{\xi\xi} - 2\eta\omega_{\xi\eta} = 0$$

and the relation

$$2R_\xi = [R^2 + 2B(\omega_\xi)\omega_{\xi\xi}.$$

It can be written in form

$$2\frac{dR}{d\Omega} = R^2 + 2B(\Omega)$$

using the notation

$$\omega_\xi = \Omega$$

Proposition 2 *In case $h \neq 0$ and $g = g(a, c)$ the system (3) is equivalent the equation*

$$\Theta_a(\frac{\Theta_a}{\Theta_c})_{ccc} - \Theta_c(\frac{\Theta_a}{\Theta_c})_{acc} = 1 \tag{6}$$

where

$$g = -\frac{\Theta_a}{\Theta_c} \quad h_c = \frac{1}{\Theta_c}$$

To integrate this equation we use the presentation

$$c = \Omega(\Theta, a)$$

From the relations

$$1 = \Omega_\Theta \Theta_c, \quad 0 = \Omega_\Theta \Theta_a + \Omega_c$$

we get

$$\Theta_c = \frac{1}{\Omega_\Theta}, \quad \Theta_a = -\frac{\Omega_a}{\Omega_\Theta}$$

and

$$\frac{\Omega_a}{\Omega_\Theta}(\Omega_a)_{ccc} + \frac{1}{\Omega_\Theta}(\Omega_a)_{cca} = 1$$

Now we get

$$\Omega_{ac} = \frac{\Omega_{a\Theta}}{\Omega_\Theta} = (\ln \Omega_\Theta)_a = K, \quad \Omega_{acc} = \frac{K_\Theta}{\Omega_\Theta},$$

$$\Omega_{accc} = (\frac{K_\Theta}{\Omega_\Theta})_\Theta \frac{1}{\Omega_\Theta}, \quad (\Omega_{acc})_a = (\frac{K_\Theta}{\Omega_\Theta})_a - \frac{\Omega_a}{\Omega_\Theta}(\frac{K_\Theta}{\Omega_\Theta})_\Theta$$

As result the equation (6) take the form

$$\left[\frac{(\ln \Omega_\Theta)_{a\Theta}}{\Omega_\Theta}\right]_a = \Omega_\Theta \tag{7}$$

and can be integrate under the substitution

$$\Omega(\Theta, a) = \Lambda_a$$

So we get the equation

$$\Lambda_{\Theta\Theta} = \frac{1}{6}\Lambda_\Theta^3 + \alpha(\Theta)\Lambda_\Theta^2 + \beta(\Theta)\Lambda(\Theta) + \gamma(\Theta) \tag{8}$$

with arbitrary coefficients α, β, γ.

This is Abel's type of equation

$$y' = A(x)y^3 + B(x)y^2 + C(x)y + D(x)$$

It can be rewriten in form

$$y' = A(y - \phi)^3 + \theta(y - \phi)^2 + \lambda(y - \phi) + \phi'$$

or

$$z' = Az^3 + \theta z^2 + \lambda z$$

Let us considere the examples.

1. $\alpha=\beta=\gamma=0$

The solution of equation (8) is

$$\Lambda = A(a) - 6\sqrt{B(a)-\frac{1}{3}\Theta}$$

and we get

$$c = A' - \frac{3B'}{\sqrt{B-\frac{1}{3}\Theta}}$$

or

$$\Theta = 3B - 27\frac{B'^2}{(c-A')^2}$$

This solution is corresponded to the equation

$$b'' = -\frac{\Theta_a}{\Theta_c} = -\frac{1}{18B'}b'^3 + \frac{A'}{6B'}b'^2 + (\frac{B''}{B'} - \frac{A'^2}{6B'})b' + A'' + \frac{A'^3}{18B'} - \frac{A'B''}{B'}$$

cubical on the first derivatives b' with arbitrary coefficients $A(a), B(a)$. This equation is equivalent to the equation

$$b''=0$$

under the point transformation.

The following example is the solution of equation (8) in form

$$g = b^{1-2\alpha}\omega[cb^{\alpha-1}]$$

Under this reduction one obtains the equation on the function $\omega(\xi = cb^{\alpha-1})$

$$\omega\omega'' - \frac{\omega'^2}{2} + (\alpha-1)\xi^2\omega'' + (2-3\alpha)\xi\omega' + 2(2\alpha-1)\omega = 0.$$

To make the new variable $\theta = \omega + (\alpha-1)\xi^2$ we obtain

$$\theta\theta'' - \frac{\theta'^2}{2} - \alpha\xi\theta' + 2\alpha\theta = 0.$$

This equation has solution in parametrical form

$$\theta = \gamma\tau E(\tau), \quad \xi = \frac{\gamma E(\tau)}{\beta}$$

where

$$E(\tau) = \exp[-\int\frac{\tau d\tau}{(\tau-1/2)^2+\alpha/\beta^2-1/4}] \tag{9}$$

where β,γ are parametrs and the explicit form of this integral depends on the value

$$\epsilon = \alpha/\beta^2 - 1/4.$$

This solution is corresponed to the family of equations

$$b'' = b^{1-2\alpha}[\theta + (1-\alpha)\xi^2]$$

and (1) forming dual paar.

The values of coefficients $a_i(x,y,\alpha,\beta,\gamma)$ in corresponding equation (1) can be change radicaly at the variation of parameters as it is showed the calculation of integral (10).

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